**Material Model 10: Elastic-Plastic-Hydrodynamic**

For completeness we give the entire derivation of this constitutive model based on radial return plasticity.

The pressure, \( p \), deviatoric strain rate, \( \dot{\varepsilon}'_i \), deviatoric stress rate, \( \dot{s}_i \), volumetric strain rate, and \( \dot{\varepsilon}_v \), are defined in Equation (19.10.1):

\[
p = -\frac{1}{3} \sigma_y \delta_i \quad \dot{\varepsilon}'_i = \dot{\varepsilon}_i = \frac{1}{3} \dot{\varepsilon}_v
\]

\[
s_i = \sigma_i + p \delta_i \quad \dot{\varepsilon}_v = \dot{\varepsilon}_i \delta_i \quad (19.10.1)
\]

\[
s_{ij}^V = 2\mu \dot{\varepsilon}'_i = 2G\dot{\varepsilon}'_i
\]

The Jaumann rate of the deviatoric stress, \( s_{ij}^V \), is given by:

\[
s_{ij}^V = \dot{s}_{ij} - s_{ij} Q_{pj} - s_{jp} Q_{pi} \quad (19.10.2)
\]

First we update \( s^n_{ij} \) to \( s^{n+1}_{ij} \) elastically

\[
*_{s_{ij}^{n+1}} = s^n_{ij} + s_{ij} Q_{pj} + s_{jp} Q_{pi} + 2G\dot{\varepsilon}'_i dt = \frac{s^n_{ij} + R_{ij} + 2G\dot{\varepsilon}'_i dt}{s^n_{ij}} \quad (19.10.3)
\]

where the left superscript, *, denotes a trial stress value. The effective trial stress is defined by

\[
s^* = \left( \frac{3}{2} \frac{s^{n+1}_{ij} + s^{n+1}_{ij}}{s^*} \right)^{\frac{1}{2}} \quad (19.10.4)
\]

and if \( s^* \) exceeds yield stress \( \sigma_y \), the Von Mises flow rule:

\[
\phi = \frac{1}{2} s_{ij} s_{ij} - \frac{\sigma_y^2}{3} \leq 0 \quad (19.10.5)
\]

is violated and we scale the trial stresses back to the yield surface, i.e., a radial return

\[
s_{ij}^{n+1} = \frac{\sigma_y}{s^*} s_{ij}^{n+1} = m s_{ij}^{n+1} \quad (19.10.6)
\]
The plastic strain increment can be found by subtracting the deviatoric part of the strain increment that is elastic, \( \frac{1}{2G} \left( s_{ij}^{n+1} - s_{ij}^{e} \right) \), from the total deviatoric increment, \( \Delta \varepsilon'_{ij} \), i.e.,

\[
\Delta \varepsilon^p_{ij} = \Delta \varepsilon'_{ij} - \frac{1}{2G} \left( s_{ij}^{n+1} - s_{ij}^{e} \right)
\]

(19.10.7)

Recalling that,

\[
\Delta \varepsilon'_{ij} = \frac{s_{ij}^{n+1} - s_{ij}^{e}}{2G}
\]

(19.10.8)

and substituting Equation (19.10.8) into (19.10.7) we obtain,

\[
\Delta \varepsilon^p_{ij} = \frac{(s_{ij}^{n+1} - s_{ij}^{n+1})}{2G}
\]

(19.10.9)

Substituting Equation (19.10.6)

\[
s_{ij}^{n+1} = m \cdot s_{ij}^{n+1}
\]

into Equation (19.10.9) gives,

\[
\Delta \varepsilon^p_{ij} = \frac{(1-m)}{2G} \cdot s_{ij}^{n+1} = \frac{1-m}{2Gm} \cdot s_{ij}^{n+1} = d \lambda s_{ij}^{n+1}
\]

(19.10.10)

By definition an increment in effective plastic strain is

\[
\Delta \varepsilon^p = \left( \frac{2}{3} \Delta \varepsilon^p_{ij} \right)^{\frac{1}{2}}
\]

(19.10.11)

Squaring both sides of Equation (19.10.10) leads to:

\[
\Delta \varepsilon^p_{ij} \Delta \varepsilon^p_{ij} = \left( \frac{1-m}{2G} \right)^2 \cdot (s_{ij}^{n+1} \cdot s_{ij}^{n+1})
\]

(19.10.12)

or from Equations (19.10.4) and (19.10.11):

\[
\frac{3}{2} \Delta \varepsilon^p = \left( \frac{1-m}{2G} \right)^2 \frac{2}{3} s^{2} s^{2}
\]

(19.10.13)

Hence,
where we have substituted for \( m \) from Equation (19.10.6)

\[
m = \frac{\sigma_y}{s^*}
\]

If isotropic hardening is assumed then:

\[
\sigma_y^{n+1} = \sigma_y^n + E^p \Delta \varepsilon^p
\]  \hspace{1cm} (19.10.15)

and from Equation (19.10.14)

\[
\Delta \varepsilon^p = \frac{(s^* - \sigma_y^{n+1})}{3G} = \frac{(s^* - \sigma_y^n - E^p \Delta \varepsilon^p)}{3G}
\]  \hspace{1cm} (19.10.16)

Thus,

\[
(3G + E^p) \Delta \varepsilon^p = (s^* - \sigma_y^n)
\]

and solving for the incremental plastic strain gives

\[
\Delta \varepsilon^p = \frac{(s^* - \sigma_y^n)}{(3G + E^p)}
\]  \hspace{1cm} (19.10.17)

The algorithm for plastic loading can now be outlined in five simple stress. If the effective trial stress exceeds the yield stress then

1. Solve for the plastic strain increment:

\[
\Delta \varepsilon^p = \frac{(s^* - \sigma_y^n)}{(3G + E^p)}
\]

2. Update the plastic strain:

\[
\varepsilon^{n+1} = \varepsilon^n + \Delta \varepsilon^p.
\]

3. Update the yield stress:

\[
\sigma_y^{n+1} = \sigma_y^n + E^p \Delta \varepsilon^p
\]
4. Compute the scale factor using the yield strength at time \( n + 1 \):

\[
m = \frac{\sigma_y^{n+1}}{s}
\]

5. Radial return the deviatoric stresses to the yield surface:

\[
s_{0}^{n+1} = m \cdot s_{0}^{n+1}
\]

**Material Model 11: Elastic-Plastic With Thermal Softening**

Steinberg and Guinan [1978] developed this model for treating plasticity at high strain rates (10\(^5\) s\(^{-1}\)) where enhancement of the yield strength due to strain rate effects is saturated out.

Both the shear modulus \( G \) and yield strength \( \sigma_y \) increase with pressure but decrease with temperature. As a melt temperature is reached, these quantities approach zero. We define the shear modulus before the material melts as

\[
G = G_0 \left[ 1 +bp\sqrt{\gamma} - h \left( \frac{E-E_c}{3R'} - 300 \right) e^{\frac{-fL}{E_m-E}} \right] \quad (19.11.1)
\]

where \( G_0, b, h, \) and \( f \) are input parameters, \( E_c \) is the cold compression energy:

\[
E_c (X) = \int_0^X p(1-x) - \frac{900}{1-x} \exp \left( \frac{ax}{(1-x)(\gamma-a)} \right) dx, \quad (19.11.2)
\]

where

\[
X = 1-V, \quad (19.11.3)
\]

and \( E_m \) is the melting energy:

\[
E_m (X) = E_c (X) + 3R'T_m (X) \quad (19.11.4)
\]

which is a function of the melting temperature \( T_m (X) \):

\[
T_m (X) = \frac{T_{m_0} \exp (2aX)}{(1-X)^2(\gamma-a-\gamma)} \quad (19.11.5)
\]

and the melting temperature \( T_{m_0} \) at \( \rho = \rho_0 \). The constants \( \gamma_0 \) and \( a \) are input parameters. In the above equation, \( R' \) is defined by