Material Model 27: Incompressible Mooney-Rivlin Rubber

The Mooney-Rivlin material model is based on a strain energy function, \( W \), as follows

\[
W = A(I_1 - 3) + B(I_2 - 3) + C\left(\frac{1}{I_3} - 1\right) + D(I_3 - 1)^2
\]  

(19.27.1)

A and B are user defined constants, whereas \( C^* \) and D are related to A and B as follows

\[
C = \frac{1}{2} A + B \quad \text{(19.27.2)}
\]

\[
D = \frac{A(5\nu - 2) + B(11\nu - 5)}{2(1 - 2\nu)} \quad \text{(19.27.3)}
\]

The derivation of the constants \( C \) and \( D \) is straightforward [Feng, 1993] and is included here since we were unable to locate it in the literature. The principal components of Cauchy stress, \( \sigma_i \), are given by [Ogden, 1984]

\[
J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} \quad \text{(19.27.4)}
\]

For uniform dilation

\[
\lambda_1 = \lambda_2 = \lambda_3 = \lambda \quad \text{(19.27.5)}
\]

thus the pressure, \( p \), is obtained (please note the sign convention),

\[
p = \sigma_1 = \sigma_2 = \sigma_3 = \frac{2}{\lambda^3} \left( \lambda^2 \frac{\partial W}{\partial I_1} + 2\lambda^4 \frac{\partial W}{\partial I_2} + \lambda^6 \frac{\partial W}{\partial I_3} \right) \quad \text{(19.27.6)}
\]

The relative volume, \( V \), can be defined in terms of the stretches as:

\[
V = \lambda^3 = \frac{\text{new volume}}{\text{old volume}} \quad \text{(19.27.7)}
\]

For small volumetric deformations the bulk modulus, \( K \), can be defined as the ratio of the pressure over the volumetric strain as the relative volume approaches unity:

\[
K = \lim_{V \to 1} \left( \frac{p}{V - 1} \right) \quad \text{(19.27.8)}
\]

* Please observe the difference between the constant, \( C \), and the right Cauchy Green tensor \( C \), which will be denoted either by boldface tensorial notation or through its tensorial components, \( C_{ij} \) throughout the report.
The partial derivatives of $W$ lead to:

\[
\begin{align*}
\frac{\partial W}{\partial I_1} &= A \\
\frac{\partial W}{\partial I_2} &= B \\
\frac{\partial W}{\partial I_3} &= -2CI_3 + 2D(I_3 - 1) = -2C\lambda^{18} + 2D(\lambda^6 - 1)
\end{align*}
\]

\[p = \frac{2}{\lambda^3} \left\{ A\lambda^2 + 2\lambda^4 B + \lambda^6 \left[ -2C\lambda^{18} + 2D(\lambda^6 - 1) \right] \right\} = \frac{2}{\lambda^3} \left\{ A\lambda^2 + 2\lambda^4 B - 2C\lambda^{12} + 2D(\lambda^{12} - \lambda^6) \right\}
\]

In the limit as the stretch ratio approaches unity, the pressure must approach zero:

\[
\lim_{\lambda \to 1} p = 0 \quad (19.27.10)
\]

Therefore, $A + 2B - 2C = 0$ and

\[
\therefore C = 0.5A + B \quad (19.27.11)
\]

To solve for $D$ we note that:

\[
K = \lim_{V \to 1} \left( \frac{p}{V - 1} \right) = \lim_{\lambda \to 1} \frac{2}{\lambda^3} \left\{ A\lambda^2 + 2\lambda^4 B - 2C\lambda^{12} + 2D(\lambda^{12} - \lambda^6) \right\} = \frac{2}{\lambda^3} \left\{ A\lambda^2 + 2\lambda^4 B - 2C\lambda^{12} + 2D(\lambda^{12} - \lambda^6) \right\}
\]

\[
= 2 \lim_{\lambda \to 1} \frac{A\lambda^2 + 2\lambda^4 B - 2C\lambda^{12} + 2D(\lambda^{12} - \lambda^6)}{\lambda^6 - \lambda^3}
\]

\[
= 2 \lim_{\lambda \to 1} \frac{2A\lambda + 8\lambda^3 B + 24C\lambda^{-13} + 2D(12\lambda^{11} - 6\lambda^5)}{6\lambda^5 - 3\lambda^2}
\]

\[
= \frac{2}{3} (2A + 8B + 24C + 12D) = \frac{2}{3} (14A + 32B + 12D)
\]

We therefore obtain:
The invariants $I_1$-$I_3$ are related to the right Cauchy-Green tensor $C$ as

$$I_1 = C_{ii}$$  \hspace{1cm} (19.27.15)$$

$$I_2 = \frac{1}{2} C_{ii}^2 - \frac{1}{2} C_{ij} C_{ji}$$  \hspace{1cm} (19.27.16)$$

$$I_3 = \text{det}(C_{ij})$$  \hspace{1cm} (19.27.17)$$

The second Piola-Kirchhoff stress tensor, $S$, is found by taking the partial derivative of the strain energy function with respect to the Green-Lagrange strain tensor, $E$.

$$S_{ij} = \frac{\partial W}{\partial E_{ij}} = 2 \frac{\partial W}{\partial C_{ij}} = 2 \left( A \frac{\partial I_1}{\partial C_{ij}} + B \frac{\partial I_2}{\partial C_{ij}} + 2D(I_3 - 1) \frac{\partial I_3}{I_3^2} \frac{\partial I_3}{\partial C_{ij}} \right)$$  \hspace{1cm} (19.27.18)$$

The derivatives of the invariants $I_1$-$I_3$ are

$$\frac{\partial I_1}{\partial C_{ij}} = \delta_{ij}$$

$$\frac{\partial I_2}{\partial C_{ij}} = I_1 \delta_{ij} - C_{ij}$$  \hspace{1cm} (19.27.19)$$

$$\frac{\partial I_3}{\partial C_{ij}} = I_3 C_{ij}^{-1}$$

Inserting Equation (19.27.19) into Equation (19.27.18) yields the following expression for the second Piola-Kirchhoff stress:

$$S_{ij} = 2A \delta_{ij} + 2B \left( I_1 \delta_{ij} - C_{ij} \right) - 4C \frac{1}{I_3^2} C_{ij}^{-1} + 4D(I_3 - 1)I_3 C_{ij}^{-1}$$  \hspace{1cm} (19.27.20)$$

Equation (19.27.20) can be transformed into the Cauchy stress by using the push forward operation.
\[ \sigma_{ij} = \frac{1}{J} F_{ik} S_{kl} F_{jl} \quad (19.27.21) \]

where \( J = \det(F_{ij}) \).

### 19.27.1 Stress Update for Shell Elements

As a basis for discussing the algorithmic tangent stiffness for shell elements in Section 19.27.3, the corresponding stress update as it is done in LS-DYNA is shortly recapitulated in this section. When dealing with shell elements, the stress (as well as constitutive matrix) is typically evaluated in corotational coordinates after which it is transformed back to the standard basis according to

\[
\sigma_{ij} = R_{ik} R_{jl} \hat{\sigma}_{kl}.
\]

Here \( R_{ij} \) is the rotation matrix containing the corotational basis vectors. The so-called corotated stress \( \hat{\sigma}_{ij} \) is evaluated using Equation 19.27.21 with the exception that the deformation gradient is expressed in the corotational coordinates, i.e.,

\[
\hat{\sigma}_{ij} = \frac{1}{J} \hat{F}_{ik} S_{kl} \hat{F}_{jl} \quad (19.27.22)
\]

where \( S_{ij} \) is evaluated using Equation (19.27.20). The corotated deformation gradient is incrementally updated with the aid of a time increment \( \Delta t \), the corotated velocity gradient \( \hat{L}_{ij} \), and the angular velocity \( \hat{\Omega}_{ij} \) with which the embedded coordinate system is rotating,

\[
\hat{F}_{ij} = (\delta_{ik} + \Delta t \hat{L}_{ik} - \Delta t \hat{\Omega}_{ik}) \hat{F}_{kj} \quad (19.27.23)
\]

The primary reason for taking a corotational approach is to facilitate the maintenance of a vanishing normal stress through the thickness of the shell, something that is achieved by adjusting the corresponding component of the corotated velocity gradient \( \hat{L}_{33} \) accordingly. The problem can be stated as to determine \( \hat{L}_{33} \) such that when updating the deformation gradient through Equation (19.27.23) and subsequently the stress through Equation (19.27.22), \( \hat{\sigma}_{33} = 0 \). To this end, it is assumed that

\[
\hat{L}_{33} = \alpha (\hat{L}_{11} + \hat{L}_{22}),
\]

for some parameter \( \alpha \) that is determined in the following three step procedure. In the first two steps, \( \alpha = 0 \) and \( \alpha = -1 \), respectively, resulting in two trial normal stresses \( \hat{\sigma}_{33}^{(0)} \) and \( \hat{\sigma}_{33}^{(-1)} \).
Then it is assumed that the actual normal stress depends linearly on \( \alpha \), meaning that the latter can be determined from

\[
0 = \sigma_{33}^{(\alpha)} = \sigma_{33}^{(0)} + \alpha(\sigma_{33}^{(0)} - \sigma_{33}^{(-1)}) .
\]

In LS-DYNA, \( \alpha \) is given by

\[
\alpha = \begin{cases} 
\frac{\hat{\sigma}_{33}^{(0)}}{\hat{\sigma}_{33}^{(-1)} - \hat{\sigma}_{33}^{(0)}}, & \text{if } |\hat{\sigma}_{33}^{(-1)} - \hat{\sigma}_{33}^{(0)}| \geq 10^{-4}, \\
-1, & \text{otherwise}
\end{cases}
\]

and the stresses are determined from this value of \( \alpha \). Finally, to make sure that the normal stress through the thickness vanishes, it is set to 0 (zero) before exiting the stress update routine.

19.27.2 Derivation of the Continuum Tangent Stiffness

This section will describe the derivation of the continuum tangent stiffness for the Mooney-Rivlin material. For solid elements, the continuum tangent stiffness is chosen in favor of an algorithmic (consistent) tangential modulus as the constitutive equation at hand is smooth and a consistent tangent modulus is not required for good convergence properties. For shell elements however, this stiffness must ideally be modified in order to account for the zero normal stress condition. This modification, and its consequences, are discussed in the next section.

The continuum tangent modulus in the reference configuration is per definition,

\[
E_{ijkl}^{PK} = \frac{\partial S_{ij}}{\partial E_{kl}} = 2 \frac{\partial S_{ij}}{\partial C_{kl}}
\]

(19.27.24)

Splitting up the differentiation of Equation (19.27.20) we get

\[
\frac{\partial (I_3 \delta_{ij} - C_{ij})}{\partial C_{kl}} = \delta_{kl} \delta_{ij} - \frac{1}{2} \left( \delta_{ki} \delta_{jl} + \delta_{kj} \delta_{il} \right)
\]

\[
\frac{\partial \left( \frac{1}{I_3} C_{ij}^{-1} \right)}{\partial C_{kl}} = - \frac{2}{I_3^2} C_{kl}^{-1} C_{ij}^{-1} - \frac{1}{2 I_3^2} \left( C_{ij}^{-1} C_{il}^{-1} + C_{ij}^{-1} C_{ik}^{-1} \right)
\]

(19.27.25)

\[
\frac{\partial \left( I_3 (I_3 - 1) C_{ij}^{-1} \right)}{\partial C_{kl}} = I_3 (2I_3 - 1) C_{kl}^{-1} C_{ij}^{-1} - \frac{1}{2} I_3 (I_3 - 1) (C_{ij}^{-1} C_{il}^{-1} + C_{ij}^{-1} C_{ik}^{-1})
\]

Since LS-DYNA needs the tangential modulus for the Cauchy stress, it is a good idea to transform the terms in Equation (19.27.25) before summing them up. The push forward operation for the fourth-order tensor \( E_{ijkl}^{PK} \) is...
\[ E_{ijkl}^{TC} = \frac{1}{J} F_{ia} F_{jb} F_{kc} F_{ld} E_{abcd} \] (19.27.26)

Since the right Cauchy-Green tensor is \( C = F^T F \) and the left Cauchy-Green tensor is \( b = F F^T \), and the determinant and trace of the both stretches are equal, the transformation is in practice carried out by interchanging

\[ C^{-1}_{ij} \rightarrow \delta_{ij}, \quad \delta_{ij} \rightarrow b_{ij} \]

The end result is then

\[ J E_{ijkl}^{TC} = 4B \left[ b_{ij} b_{ij} - \frac{1}{2} (b_{ij} b_{ij} + b_{ij} b_{ij}) \right] + \frac{4C}{I_3} \left[ 4 \delta_{ij} \delta_{kl} + \left( \delta_{ij} \delta_{kl} + \delta_{ij} \delta_{kl} \right) \right] + \\
8D I_3 \left[ (2I_3 - 1) \delta_{ij} \delta_{kl} - \frac{1}{2} (I_3 - 1) \left( \delta_{ij} \delta_{kl} + \delta_{ij} \delta_{kl} \right) \right] \] (19.27.27)

### 19.27.3 The Algorithmic Tangent Stiffness for Shell Elements

The corotated tangent stiffness matrix is given by Equation (19.27.27) with the exception that the left Cauchy-Green tensor and deformation gradient are given in corotational coordinates, i.e.,

\[ J E_{ijkl}^{TC} = 4B \left[ \hat{b}_{ij} \hat{b}_{ij} - \frac{1}{2} (\hat{b}_{ij} \hat{b}_{ij} + \hat{b}_{ij} \hat{b}_{ij}) \right] + \frac{4C}{I_3} \left[ 4 \delta_{ij} \delta_{kl} + \left( \delta_{ij} \delta_{kl} + \delta_{ij} \delta_{kl} \right) \right] + \\
8D I_3 \left[ (2I_3 - 1) \delta_{ij} \delta_{kl} - \frac{1}{2} (I_3 - 1) \left( \delta_{ij} \delta_{kl} + \delta_{ij} \delta_{kl} \right) \right] \] (19.27.28)

Using this exact expression for the tangent stiffness matrix in the context of shell elements is not adequate since it does not take into account that the normal stress is zero and it must be modified appropriately. To this end, we assume that the tangent moduli in Equation (19.27.28) relates the corotated rate-of-deformation tensor \( \hat{D}_{ij} \) to the corotated rate of stress \( \dot{\hat{\sigma}}_{ij} \),

\[ \dot{\hat{\sigma}}_{ij} = \hat{E}_{ijkl}^{TC} \hat{D}_{kl} \] (19.27.29)

Even though this is not completely true, we believe that attempting a more thorough treatment would hardly be worth the effort. The objective can now be stated as to find a modified tangent stiffness matrix \( \hat{E}_{ijkl}^{TC} \) such that

\[ \dot{\hat{\sigma}}_{ij}^{\text{alg}} = \hat{E}_{ijkl}^{TC} \hat{D}_{kl} \] (19.27.30)
where $\sigma_{ij}^\text{alg}$ is the stress as it is evaluated in LS-DYNA. The stress update, described in Section 19.27.1, is performed in a rather ad hoc way which probably makes the stated objective unachievable. Still we attempt to extract relevant information from it that enables us to come somewhat close.

An example of a modification of this tangent moduli is due to Hughes and Liu [1981] and given by

$$
\hat{E}_{ijkl}^{TCAlg} = \hat{E}_{ijkl}^{TC} - \frac{\hat{E}_{ijkl}^{TC} \hat{E}_{ijkl}^{TC}}{\hat{E}_{3333}^{TC}}.
$$

This matrix is derived by eliminating the thickness strain $\hat{D}_{33}$ from the equation $\sigma_{33}^* = 0$ in Equation (19.27.30) as an unknown. This modification is unfortunately not consistent with how the stresses are updated in LS-DYNA. When consulting Section 19.27.1, it is suggested that $\hat{D}_{33}$ instead can be eliminated from

$$
\hat{D}_{33} = \alpha(\hat{D}_{11} + \hat{D}_{22})
$$

using the $\alpha$ determined from the stress update. Unfortunately, by the time when the tangent stiffness matrix is calculated, the exact value of $\alpha$ is not known. From experimental observations however, we have found that $\alpha$ is seldom far from being equal to $-1$. The fact that $\alpha = -1$ represents incompressibility strengthen this hypothesis. This leads to a modified tangent stiffness $\hat{E}_{ijkl}^{TCAlg}$ that is equal to $\hat{E}_{ijkl}^{TC}$ except for the following modifications,

$$
\hat{E}_{ijkl}^{TCAlg} = \hat{E}_{ijkl}^{TC} - \hat{E}_{ijkl}^{TC} - \hat{E}_{ijkl}^{TC} + \hat{E}_{ijkl}^{TC},
$$

$$
\hat{E}_{ij33}^{TCAlg} = \hat{E}_{ij33}^{TCAlg} = 0, \ i \neq j
$$

To preclude the obvious singularity, a small positive value is assigned to $\hat{E}_{3333}^{TCAlg}$,

$$
\hat{E}_{3333}^{TCAlg} = 10^{-4} \left( \left| \hat{E}_{1111}^{TCAlg} \right| + \left| \hat{E}_{2222}^{TCAlg} \right| \right).
$$

As with the Hughes-Liu modification, this modification preserves symmetry and positive definiteness of the tangent moduli, which together with the stress update “consistency” makes it intuitively attractive.

**Material Model 28: Resultant Plasticity**

This plasticity model, based on resultants as illustrated in Figure 19.29.1, is very cost effective but not as accurate as through-thickness integration. This model is available only with the $C^0$ triangular, Belytschko-Tsay shell, and the Belytschko beam element since these elements, unlike the Hughes-Liu elements, lend themselves very cleanly to a resultant formulation.